B.Sc. Semester III (Honours) Examination, 2018-19

MATHEMATICS

Course ID : 32111 **Course Code : SHMTH-301C-5(T)**

Course Title : Theory of Real Functions and Introduction to Metric Space

Time: 2 Hours **Full Marks: 40**

The figures in the margin indicate full marks.
Candidates are required to give their answers in their own words as far as practicable.

1. Answer *any five* questions: 2×5=10

(a) If $\lim_{x\to a} f(x) = l$, then show that $\lim_{x\to a}|f(x)| = |l|$.

(b) Show that the function $f: [0,1] \to \mathbb{R}$ defined by

$f(x) = 1$, if x is rational

$= 0$, if x is irrational

is continuous nowhere in [0, 1]

(c) Prove that the function $f(x) = \frac{1}{x}, x \in (0,1]$ is not uniformly continuous on (0, 1].

(d) Is Rolle's theorem satisfied for $f(x) = x^2$ in [–1, 1]. Justify your answer.

(e) Show that $\log(1 + x) < x$ for $x > 0$.

(f) Expand $\tan^{-1} x$ (up to three terms).

(g) Give an example of a function f which satisfies the intermediate-value property on a closed and bounded interval $[a, b]$ but is not continuous on $[a, b]$.

(h) The function $d: R \times R \to R$ defined by $d(x, y) = |x - y|, \forall x, y \in \mathbb{R}$. Show that d is a metric on the set $\mathbb{R}$.

2. Answer *any four* questions: 5×4=20

(a) (i) If $f: S \to \mathbb{R}$ be differentiable at $c \in S$, then show that there exist $\delta > 0$ and a positive constant M such that $|f(x) - f(c)| \leq M\,|x - c|, \forall\, x \in S \cap N_\delta(c)$.

(ii) A function f is differentiable on [0, 2] and $f(0) = 0, f(1) = 2, f(2) = 1$. Prove that $f'(c) = 0$ for some c in (0, 2). 3+2=5

(b) Let $f: \mathbb{R} \to \mathbb{R}$ be a function satisfying $f(x + y) = f(x) + f(y), \forall\, x, y \in \mathbb{R}$. Show that if f is continuous at $x = a$, then f is continuous for all $x \in \mathbb{R}$. 5

(c) State and prove Cauchy's mean value theorem and hence deduce Lagrange's mean value theorem. 1+3+1=5

(d) Obtain Maclaurin's infinite series expansion of $\log(1+x), -1 < x \leq 1$. 5

(e) Examine the function $(x-3)^5(x+1)^4$ for extreme values. 5

(f) Let (X, d) be a metric space and A° denote the interior of A. Then show that $(A \cap B)^\circ = A^\circ \cap B^\circ$. Is $(A \cup B)^\circ = A^\circ \cup B^\circ$ true for any subjects A, B of X? Give reason for your answer. 3+2=5

3. Answer *any one* question: 10×1=10

(a) (i) If $f(x)$ is a function defined on a deleted neighbourhood D of a point 'a' such that $f(x) \geq 0 \;\forall\; x \in D$, then show that $\lim_{x \to a} f(x) \geq 0$, provided it exists.

(ii) If a function '*f*' is continuous in a closed interval [*a, b*], then show that '*f*' is bounded in [*a, b*]. Also show that the converse of the above is not true in general.

(iii) Let (X, d) be a metric space such that *X* contains more than one point and $A \subseteq X$. Show that a point $x \in X$ is a limit point of *A* iff every open sphere $S(x.r)$ contains infinitely many points of *A*. 3+(2+2)+3=10

(b) (i) Give an example of a pair of functions *f, g* on [*a, b*] such that both *f, g* are discontinuous on [*a, b*], but *f* + *g* is continuous on [*a, b*].

(ii) Show that the function $f(x) = \frac{1}{x^2}$ is uniformly continuous on $[a, \infty)$, where $a > 0$; but not uniformly continuous on $(0, \infty)$.

(iii) State and prove Taylor's theorem with Lagrange's form of remainder. 2+3+5=10

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